Proceeding International Conference on Religion, Science and Education (2022) **1**, 495-500

Numerical Solution of Linear Integral Equations Using Modified Block Pulse Functions

Ayyubi Ahmad
Mathematics Department, Faculty of Applied Science and Technology, Universitas Ahmad Dahlan Yogyakarta,
Jl. Ringroad Selatan, Bantul, Yogyakarta 55191, Indonesia.

Corresponding author
ayyubiahmad1@gmail.com

Abstract: A computational method based on modified block pulse functions is proposed for solving numerically the linear Volterra and Fredholm integral equations. We obtain integration operational matrix of modified block pulse functions on interval [0,1).A modified block pulse functions and their integration operational matrix can be reduced to a linear upper triangular system.Then, the problem under study is transformed to a system of linear algebraic equations which can be used to obtain an approximate solution of linear Volterra and Fredholm integral equations.Furthermore, the rate of convergence is $O(h)$ and error analysis of the proposed method are investigated. The results show that the approximate solutions have a good of efficiency and accuracy.

Keywords: Brownian Motion, Integration Operational Matrix, Linear Integral Equations, εMBPFs.

Introduction

Linear integral equations and their solutions are great importance in the fields of science and engineering. Most physical problems, such as biological applications in population dynamics and genetics, where impulses occur naturally or are caused by controls, can be modeled by an integral equation [7]. The linear Volterra and Fredholm integral equations each appear in the following form, respectively

$$f(x) = g(x) + \int_0^x k(x,y)f(y)\,dy, x \in I = [0,1), \quad (1)$$

and

$$f(x) = g(x) + \int_0^1 k(x,y)f(y)\,dy, \qquad x \in I, \quad (2)$$

where $f(x)$ is an unknown function, $g(x)$ and $k(x,y)$ are analytical function on I and $I \times I$, respectively.

Approximation theory is how functions can be approximated with simpler functions called base functions and with quantitatively characterizing the errors. One of these base functions is Block Pulse Functions (BPFs). BPFs are very common in use and it seems their convergence is weak [3]. In fact by referring to error bound of BPFs approximation for achieving double precision, number of BPFs have to be doubled which means solving systems of equations with double unknowns and double equations [6].

In recent years, different orthogonal basic functions or polynomials such as block pulse functions, Walsh functions, Fourier series, orthonormal bases, Legendre polynomials, Chebyshev polynomials, Laguerre polynomials and wavelets, were used to estimate solutions of some systems such as integral equations [2, 4]. In this study, the ε Modified Block Pulse Functions (εMBPFs) were introduced and several theorems proved for the εMBPFs method which showed that the results of this numerical extension were more accurate than the results of the block pulse extension. These functions are disjoint, orthogonal, and complete. According to the disjointness of εMBPFs, the compound form will disappear at each subinterval when multiplication, division, and some other operations are applied. The

ISSN 2829-3355 | EISSN 2828-8467

orthogonality of εMBPFs will cause the operational matrix to become a sparse matrix. The completeness of the εMBPFs guarantees that an arbitrary small mean square error can be obtained for a finite real function having only a finite number of discontinuous points in the interval $x \in [0,1)$ by increasing the number of patterns in the modified block pulse series.

We use εMBPFs and directly solve linear Volterra and Fredholm integral equations and by some examples to show the efficiency of εMBPFs.

Materials and Methods

A set of ε modified block pulse functions $\psi_i(t), i = 0,1,\ldots,m$ on the interval $[0,T)$ are defined as

$$\psi_0(t) = \begin{cases} 1 & t \in [0, h-\varepsilon) = I_0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_i(t) = \begin{cases} 1 & t \in [ih-\varepsilon, (i+1)h-\varepsilon) = I_i, \\ 0 & \text{otherwise}, \end{cases}$$

for $i = 1, 2, \ldots, m-1$, and

$$\psi_m(t) = \begin{cases} 1 & t \in [T-\varepsilon, T) = I_m, \\ 0 & \text{otherwise}. \end{cases}$$

with a positive integer value for m and $h = \frac{T}{m}$.

Similar to BPFs, the important properties of εMBPFs are as follows

- Disjointness:
$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t) & i = j, \\ 0 & i \neq j, \end{cases}$$
where $i,j = 0,\ldots,m$.
- Orthogonality:
$$\int_0^T \psi_i(t)\psi_j(t)\,dt = h\delta_{ij},$$
where $i,j = 1,\ldots,m-1$ and δ_{ij} is Kronecker delta.
- Completeness:
$$\int_0^T f^2(t)\,dt = \sum_{i=0}^{\infty} f_i^2 \|\psi_i(t)\|^2,$$
where
$$f_i = \frac{1}{\Delta(I_i)} \int_0^T f(t)\psi_i(t)\,dt, \tag{3}$$

and $\Delta(I_i)$is length of interval I_i.

Rewriting Eq. (3) in the vector form we have

$$f(t) \simeq \sum_{i=0}^{m} f_i\psi_i(t) = F^T\Psi(t) = \Psi^T(t)F,$$

in which

$$F = (f_0 f_1 \ \ldots \ f_m)^T,$$

and

$$\Psi(t) = \big(\psi_0(t)\psi_1(t) \ \ldots \ \psi_m(t)\big)^T. \tag{4}$$

Moreover, any two dimensional function $k(s,t) \in L^2([0,T_1) \times [0,T_2))$ can be expanded with respect to εMBPFs such as

$$k(s,t) = \Psi^T(s)K\Psi(t) = \Psi^T(t)K^T\Psi(s),$$

where $\Psi(s)$ and $\Psi(t)$ are m_1 and m_2 dimensional εMBPFs vectors respectively, and $K = (k_{ij}), i = 0,1,\ldots,m_1, j = 0,1,\ldots,m_2$ is the $m_1 \times m_2 \varepsilon$ modified block pulse coefficient matrix with

$$k_{ij} = \frac{1}{\Delta(I_i)\Delta(I_j)} \int_0^{T_1}\int_0^{T_2} k(s,t)\,\Psi_i(s)\Psi_j(t)\,dtds,$$

For convenience, we put $m_1 = m_2 = m$.

With defining $\Psi_{m+1}(t) = (\psi_0(t)\psi_1(t) \ldots \ \psi_m(t))^T$, we have

$$\Psi_{m+1}(t)\Psi_{m+1}^T(t) = \begin{pmatrix} \psi_0(t) & 0 & \cdots & 0 \\ 0 & \psi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_m(t) \end{pmatrix}_{(m+1)\times(m+1)}.$$

Furthermore,

$$\Psi_{m+1}^T(t)\Psi_{m+1}(t) = 1,$$

and

$$\Psi_{m+1}(t)\Psi_{m+1}^T(t)F = D_F\Psi_{m+1}(t),$$

where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_0 f_1 \ \ldots \ f_m)^T$.

Similar to BPFs,

$$\int_0^t \Psi_{m+1}(s)\,ds \simeq Q\Psi_{m+1}(t),$$

where the integration operational matrix Q of εMBPFs is given by

$$Q = \begin{pmatrix} \frac{h-\varepsilon}{2} & h-\varepsilon & \cdots & h-\varepsilon \\ 0 & \frac{h}{2} & \cdots & h \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\varepsilon}{2} \end{pmatrix}_{(m+1)\times(m+1)}.$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)\,ds \simeq \int_0^t F^T\Psi_{m+1}(s)\,ds \simeq F^T Q\Psi_{m+1}(t).$$

Results and Discussion

Volterra Integral Equations

We approximate functions $f(t)$, $g(t)$, and $k(s,t)$ in Eq. (1) with εMBPFs as follows

$$\begin{gathered} f(t) \simeq F^T\Psi(t), \\ g(t) \simeq G^T\Psi(t), \\ k(s,t) \simeq \Psi^T(s)K\Psi(t), \end{gathered} \tag{5}$$

$\Psi(t)$ is defined by (4), F, G, and K are εMBPFs coefficients of $f(t)$, $g(t)$, and $k(s,t)$, respectively. By substituting Eq. (5) in Eq. (1), we get

$$F^T\Psi(t) = G^T\Psi(t) + F^T\left(\int_0^t \Psi(s)\Psi^T(s)\,ds\right)K\Psi(t). \tag{6}$$

Let K^i be the ith row of the constant matrix K and R^i be the ith row of the integration operational matrix Q, D_{K^i} be a diagonal matrix with K^i as its diagonal entries. Assuming $m_1 = m_2$, we have

$$\begin{aligned}
&\left(\int_0^t \Psi(s)\Psi^T(s)\,ds\right)K\Psi(t) \\
&= \begin{bmatrix} R^0\Psi(t) & 0 & \cdots & 0 \\ 0 & R^1\Psi(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R^m\Psi(t) \end{bmatrix} \begin{bmatrix} K^0 \\ K^1 \\ \vdots \\ K^m \end{bmatrix} \Psi(t) \\
&= \begin{bmatrix} R^0\Psi(t)K^0\Psi(t) \\ R^1\Psi(t)K^1\Psi(t) \\ \vdots \\ R^m\Psi(t)K^m\Psi(t) \end{bmatrix} \\
&= \begin{bmatrix} R^0\Psi(t)\Psi^T(t){K^0}^T \\ R^1\Psi(t)\Psi^T(t){K^1}^T \\ \vdots \\ R^m\Psi(t)\Psi^T(t){K^m}^T \end{bmatrix} \\
&= \begin{bmatrix} R^0 D_{K^0} \\ R^1 D_{K^1} \\ \vdots \\ R^m D_{K^m} \end{bmatrix} \Psi(t) \\
&= B\Psi(t),
\end{aligned} \tag{7}$$

where

$$B = \begin{bmatrix} k_{00}\left(\frac{h-\varepsilon}{2}\right) & 2k_{01}(h-\varepsilon) & \dots & 2k_{0m}(h-\varepsilon) \\ 0 & k_{11}\left(\frac{h}{2}\right) & \dots & 2k_{1m}(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{mm}\left(\frac{\varepsilon}{2}\right) \end{bmatrix}_{(m+1)\times(m+1)}. \tag{8}$$

With substituting relations (7) and (8) in (6)

$$F^T\Psi(t) \simeq G^T\Psi(t) + F^T B\Psi(t).$$

Then,

$$F^T(I-B) \simeq G^T.$$

So, by setting $W = (I-B)^T$ and replacing $\simeq$ by $=$, we will have

$$WF = G.$$

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficients of the unknown $f(t)$.

Fredholm Integral Equations

We approximate functions $f(x)$, $g(x)$, and $k(x,y)$ in Eq. (2) with εMBPFs as follows

$$\begin{gathered} f(x) \simeq F^T\Psi(t), \\ g(t) \simeq G^T\Psi(t), \\ k(s,t) \simeq \Psi^T(s)K\Psi(t), \end{gathered} \tag{9}$$

$\Psi(t)$ is defined by (4), F, G, and K are εMBPFs coefficients of $f(t)$, $g(t)$, and $k(s,t)$, respectively. By substituting Eq. (9) in Eq. (2), we get

$$F^T\Psi(t) = G^T\Psi(t) + F^T\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)K\Psi(t), \tag{10}$$

Assuming $m_1 = m_2$ and using the relation $\int_0^1 \Psi(s)\Psi^T(s)\,ds = hI$ we have

$$\begin{aligned} \left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)K\Psi(t) &= hIK\Psi(t) \\ &= T\Psi(t), \end{aligned} \tag{11}$$

where $T = hK$. With substituting (11) in (10)

$$F^T\Psi(t) \simeq G^T\Psi(t) + F^T T\Psi(t).$$

Then,

$$F^T(I-T) \simeq G^T.$$

So, by setting $Z = (I-T)^T$ and replacing $\simeq$ by $=$, we will have

$$ZF = G.$$

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficients of the unknown $f(t)$.

Error Analysis

In the following theorems, for simplicity we assume $T = 1$ and $h = \frac{1}{m}$.

Theorem 1. *If* $\hat{f}_m(t) = \sum_{i=0}^{m} f_i\, \psi_i(t)$ *and* $f_i = \frac{1}{\Delta(I_i)} \int_0^1 f(t)\, \psi_i(t) dt, i = 0, \dots, m$ *then:*

i. $\delta = \int_0^1 \left(f(t) - \sum_{i=0}^{m} f_i\, \psi_i(t)\right)^2 dt,$ *achieves its minimum value.*

ii. $\{\hat{f}_m(t)\}$ *approaches* $f(t)$ *pointwise.*

iii. $\int_0^1 f^2(t)\, dt = \sum_{i=0}^{\infty} f_i^2\, \|\psi_i\|^2.$

[Proof]. Proof is like similar theorem in [1] but intervals of integration have to redefine as $I_i, i = 0, \dots, m$ in (3.1). ∎

Theorem 2. *Assume:*

i. $f(t)$*is continuous and differentiable in* $[-h, 1+h]$ *with bounded derivative, that is* $|f'(t)| < M$.

ii. $\hat{f}_{\frac{ih}{k}}(t), i = 0, \dots, k-1,$ *are correspondingly BPFs,*$\frac{h}{k}$ *MBPFs … ,* $\frac{(k-1)h}{k}$ *MBPFs expansions of* $f(t)$ *base on* $m + 1\varepsilon$*MBPFs over interval* $[0,1)$.

iii. $\bar{f}(t) = \frac{1}{k}\sum_{i=0}^{k-1} \hat{f}_{\frac{ih}{k}}(t).$

Then

$$\left\|f(t) - \hat{f}_{\frac{ih}{k}}(t)\right\| = \mathcal{O}(h),$$

and

$$\|f(t) - \bar{f}(t)\| = \mathcal{O}\left(\frac{h}{k}\right) \text{ in } [h, 1-h].$$

[Proof]. Trapezoidal rule for integral is

$$\int_a^b f(t)\, dt = \frac{b-a}{2}\big(f(a) + f(b)\big) - \frac{(b-a)^3 f''(\eta)}{12}$$

$$= \frac{b-a}{2}\big(f(a) + f(b)\big) + E, \qquad (12)$$

where $\eta \in [a, b]$ and E is error of integration. Suppose $t_i = \frac{i}{m} = ih$ and $I_i = [t_{i-1}, t_i]$. The representation error when $f(t)$ is represented by a series of BPFs over every subinterval $\left[t_i, t_i + \frac{h}{k}\right], i = 0, \dots, m-1$ is

$$e_i(t) = f(t) - f_i \psi_i(t) = f(t) - f_i,$$

where $f_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(t)\, dt$. From (12),

$$f_i = \frac{1}{2}\big(f(t_i) + f(t_i + h)\big) + E.$$

It is obvious that if $f(t) = C$ (constant), then $e_i(t) = 0$. So, this error is computed for $f(t) = t$ in interval $\left[t_i, t_i + \frac{h}{k}\right], i = 1, \dots, m-1.$

For this function $E = 0$, so

$$e_i(t)_{\left[t_i, t_i + \frac{h}{k}\right]} = |t - f_i|$$

$$= \left|t - \frac{t_i + t_{i+1}}{2}\right|$$

$$= \left|t - \left(t_i + \frac{h}{2}\right)\right|$$

$$\leq \frac{h}{2}.$$

Then this error with BPFs is $\frac{h}{2}M$.

Similarly, the error when $f(t)$ is represented in a series of εMBPFs over every subinterval $\left[t_i, t_i + \frac{h}{k}\right]$ is

$$e_i(t)_{\left[t_i, t_i + \frac{h}{k}\right]} = \left|t - \left(\frac{\sum_{j=0}^{k-1}\left(t_i - \left(\frac{jh}{k}\right) + t_{i+1} - \left(\frac{jh}{k}\right)\right)}{2k}\right)\right|$$

$$= \left|t - \left(\frac{\sum_{j=0}^{k-1}\left(t_i - \left(\frac{jh}{k}\right) + t_i + h - \left(\frac{jh}{k}\right)\right)}{2k}\right)\right|$$

$$= \left|t - \left(t_i + \frac{h}{2}\right) - \frac{(k-1)h}{2k}\right| \leq \frac{h}{2k}.$$

So, the error with εMBPFs is $\frac{h}{2k}M$.

For I_0 in $\left[0, \frac{h}{k}\right]$ we have

$$e_i(t)_{\left[0, \frac{h}{k}\right]} = \left|t - \sum_{j=0}^{k-1} \frac{h - \left(\frac{jh}{k}\right)}{2k}\right|$$

$$= \left|t - \left(\frac{h}{2} - \frac{(k-1)h}{4k}\right)\right|$$

$$= \left|t - \left(\frac{h}{4} + \frac{h}{4k}\right)\right|$$

$$= \mathcal{O}\left(\frac{h}{4}\right).$$

So, the error is $\mathcal{O}\left(\frac{h}{4}\right)$also for I_n.

Now,

$$\|e_i(t)\|^2 = \int_{t_i}^{t_i + \frac{h}{k}} |e_i(t)|^2\, dt$$

$$= \int_{t_i}^{t_i + \frac{h}{k}} \frac{h^2}{4k^2} M^2\, dt$$

$$= \frac{h^3}{4k^3} M^2,$$

$$\|e\|^2 = \int_0^1 e^2(t)\, dt$$

$$= \int_0^1 \left(\sum_{i=1}^{m}\sum_{j=0}^{k-1} e_i(t)\right)^2 dt$$

$$= \sum_{i=1}^{m} \sum_{j=0}^{k-1} \int_0^1 e_i^2(t)\, dt$$
$$= \sum_{i=1}^{m} \sum_{j=0}^{k-1} \|e_i(t)\|^2$$
$$= \frac{1}{h}.k.\frac{h^3}{4k^3} M^2$$
$$= \frac{h^2}{4k^2} M^2.$$

We define the representation error between $f(s,t)$ and its 2D-εMBPFs expansion f_{ij} over every subregion D_{ij}, is defined as

$$e_{ij}(s,t) = f(s,t) - f_{ij},$$

where $D_{ij} := \left\{(s,t)\,\middle|\, t_i \le s \le t_i + \frac{h}{k}, t_j \le t \le t_j + \frac{h}{k}\right\}$. With Taylor's expansion and similarity to the above discussion,

$$\|e(s,t)\| = \frac{h}{2k} M. \qquad \blacksquare$$

Theorem 3. *Assume that*

i. $P(\omega \in \Omega : \|u(\omega,t)\| < C) = 1.$

ii. $\|k_i\| < C, i = 1,2.$

Then

$$\sup(E(\|(u - \bar{u})\|)^2)^{\frac{1}{2}} = O\left(\frac{h}{k}\right), \quad t \in [h, 1-h].$$
$0 \le t \le T$

[Proof]. For a complete proof see [5]. $\blacksquare$

Numerical Examples

Example 1. Consider the following linear Volterra integral equation,

$$f(t) = 5t^3 - t^5 + \int_0^t sf(s)\, ds,$$

where $s, t \in [0,1)$. The exact solution is $f(t) = 5t^3$.

The exact and approximation solution are shown in Table 1. The curves in Fig. 1 represents a trajectory of the exact and approximation solution computed by the presented method.

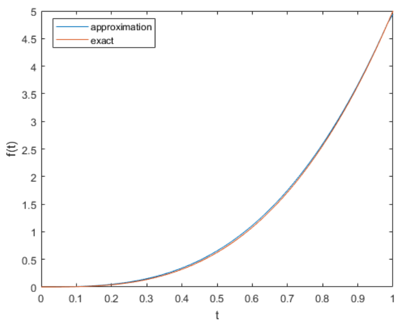

Figure 1. The trajectory of the exact and approximation solution of example 1.

Table 1. The exact and approximate solution of Example 1 for $m = 65$.

t	Exact	Approximation
0	0	0.000001112
0.1	0.005	0.00757
0.2	0.04	0.04811
0.3	0.135	0.1505
0.4	0.32	0.3433
0.5	0.625	0.6545
0.6	1.08	1.114
0.7	1.715	1.75
0.8	2.56	2.59
0.9	3.645	3.665
1	5	4.942

Example 2. Consider the following linear Fredholm integral equation,

$$f(t) = t + e^t - \frac{4}{3} + \int_0^1 sf(s)\, ds,$$

where $t \in [0,1)$. The exact solution is $f(t) = t + e^t$.

The exact and approximation solution are shown in Table 2. The curves in Fig. 2 represents a trajectory of the exact and approximation solution computed by the presented method.

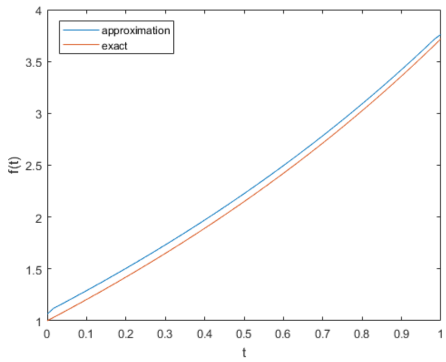

Figure 2. The trajectory of the exact and approximation solution of example 2.

Table 2. The exact and approximatesolution of Example 2 for $m = 65$.

t	Exact	Approximation
0	1	1.064
0.1	1.205171	1.291
0.2	1.421403	1.505
0.3	1.649859	1.732
0.4	1.891825	1.972
0.5	2.148721	2.226
0.6	2.422119	2.496
0.7	2.713753	2.784
0.8	3.025541	3.092
0.9	3.359603	3.422
1	3.718282	3.761

Conclusions

The εMBPFs and their integration operational matrix are used to obtain the solution of linear Volterra and Fredholm integral equations. The present method reduces a linear Volterra and Fredholm integral equations into a system of algebraic equations. The matrices Q have many zeros and the method is computationally very attractive. Its applicability and accuracy of the proposed method was checked on some examples. The results show that the approximate solutions of the proposed method have a good of efficiency and accuracy.

References

Jiang, Z. H. and W. Schaufelberger. 1992. Block Pulse Functions and Their Applications in Control Systems. Springer, Germany.

Khodabin, M., K. Maleknejad, M. Rostami, and M. Nouri. 2012. Numerical Approach for Solving Stochastic Volterra-Fredholm Integral Equations by Stochastic Operational Matrix. Computers and Mathematics with Applications 64: 1903-1913.

Maleknejad, K. and B. Rahimi. 2011. Modification of Block Pulse Functions and Their Application to Solve Numerically Volterra Integral Equation of The First Kind. Commun Nonlinear Sci Numer Simulat 16: 2469-2477.

Maleknejad, K. and Mahmoudi Y. 2004. Numerical Solution of Linear Fredholm Integral Equations by Using Hybrid Taylor and Block Pulse Functions. Appl Math Comput 149: 799-806.

Maleknejad, K., M. Khodabin, and F. Hosseini Shekarabi. 2014. Modified Block Pulse Functions for Numerical Solution of Stochastic Volterra Integral Equations. Journal of Applied Mathematics 4: 1-10.

Maleknejad, K., Sohrabi S., and Baranji B. 2010. Application of 2D-BPFs to Nonlinear Integral Equations. Commun Nonlinear Sci Numer Simulat 15: 527-535.

Mirzaee, Farshid. 2016. Numerical Solution of System of Linear Integral Equations Via Improvement of Block-Pulse Functions. Journal of Mathematical Modeling 4(2): 133-159.