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The Use of Modified Block Pulse Functions for Solving the Stochastic Volterra-Fredholm Integral Equations

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Abstract. A computational method based on modified block pulse functions is proposed for solving numerically stochastic VolterraFredholm integral equations. We obtain stochastic integration operational matrix of modified block pulse functions on interval *[0,1)*. Amodified block pulse functions and their stochastic integration operational matrix can be reduced to a linear upper triangular system. Then, the problem under study is transformed to a system of linear algebraic equations which can be used to obtain an approximate solution of stochastic Volterra-Fredholm integral equations. Furthermore, the rate of convergence is *(h)* and error analysis of the proposed method are investigated. The results show that the approximate solutions have a good of efficiency and accuracy.

Keywords: Brownian Motion, Itô Integral, Stochastic Integration Operational Matrix, SV-FIEs, εMBPFs.

Running title: The Stochastic Volterra-Fredholm Integral Equations.

INTRODUCTION

The stochastic integral equations are often used as mathematical models in engineering, biology, chemistry, economics, and epidemiology. These systems are dependent on a noise source and a Gaussian white noise governed by certain probability laws (Khodabin et al. 2012). So that modeling such phenomena naturally requires the use of various stochastic differential equations or in more complicated cases like stochastic Volterra-Fredholm integral equations (Khodabin *et al*. 2012; Maleknejad *et al*. 2014). Most SV-FIEs do not have analytic solutions and it is important to find approximate solutions by using some numerical methods.

Recently, different orthogonal basis functions such as block pulse functions, Walsh functions, Fourier series, and orthogonal polynomials were used to estimate solutions of functional equations (Khodabin *et al*. 2012). In this paper, we use modified block pulse functions and stochastic integration operational matrix to consider the following linear stochastic Volterra-Fredholm integral equation:

$$X(t) = f(t) + \int_{\alpha}^{\beta} k_1(s,t)X(s)ds + \int_{0}^{t} k_2(s,t)X(s)ds + \int_{0}^{t} k(s,t)X(s)dB(s), t \in [0,T] \quad (1)$$

where $X(t)$, $f(t)$, $k_1(s,t)$, $k_2(s,t)$, $k_3(s,t)$ for $s,t \in [0, T]$ are the stochastic processes defined on the same probability space (Ω, Ƒ, P), $X(t)$ is unknown, $B(t)$ is a Brownian motion process, and $\int_0^t k_3(s,t)\, X(s)dB(s)$ is the It ô integral. In order to obtain an approximate solution for Eq. (1) based on modified block pulse functions we derive a new stochastic integration operational matrix and reduce our problem to solving a system of linear algebraic equations. Moreover, a new technique for computation of the linear terms in such equations is presented. Furthermore, convergence analysis of modified block pulse functions is investigated. We also demonstrate the efficiency and accuracy of the proposed method.

MATERIALS AND METHODS

Stochastic Calculus

Brownian Motion

A real-valued stochastic process (t), $t \in [0, T)$ is called Brownian motion if it satisfies the following properties:

- B(0) = 0.
- $B(t)$ has independent increments for $0 \le t_1 < t_2 < \dots < t_n$, random variable

$(t_1), (t_2) - (t_1), \dots, B(t_n) - B(t_{n-1})$
is independent.

- For all $0 \le s < t$, random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, that is for all $a < b$

$$P\{a \le B(t) - B(s) \le b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_{a}^{b} \frac{x^2}{e^{2(t-s)}} dx$$

- Almost all $B(t)$, $t \ge 0$ is continuous functions of t with probability 1.

Let $\{\mathcal{N}t\}$, $t \ge 0$ be an increasing family of σ-algebras of subsets of Ω. A process $g\,(t, \omega)$: $[0, \infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \ge 0$ the function $\omega \to (t, \omega)$ is $\mathcal{N}_t$-measurable.

Let $\mathcal{V} = (S, T)$ be the class of functions (t, ω): $[0, \infty) \to \Omega \times \mathbb{R}$ such that:

- The function $(t, \omega) \mapsto (t, \omega)$ is $\mathfrak{B} \times$ Ƒ-measurable, where $\mathfrak{B}$ denotes the borel algebra on $[0, \infty)$ and Ƒ is the σ-algebra on Ω.
- f is adapted to $Ƒ_t$, where $Ƒ_t$ is the σ-algebra generated by the random variables $B(s)$, $s \le t$.
- $E\,(\int_S^T ƒ^2(t, \omega)dt) < \infty$.

The Itô Integral.
Let $f \in v(S, T)$, then the Itô integral of f is defined by

$$I[f](\omega) = \int_S^T f(t,\omega)\, dB_t(\omega)$$

$$= \lim_{n \to \infty} \int_S^T \varphi_n(t,\omega) dB_t(\omega), \qquad (\text{lim in } L^2(P))$$

where φ_n is a sequence of elementary functions such that

$$E\left(\int_S^T \left(f(t,\omega) - \varphi_n(t,\omega)\right)^2 dt\right) \to 0, \quad \text{as } n \to \infty.$$

ε Modified Block Pulse Functions.
A set of ε modified block pulse functions $\psi_i(t)$, $i = 0,1, \dots, m$ on the interval $[0, T)$ are defined as

$$\psi_0(t) = \begin{cases} 1 & t \in [0, h\text{-}\varepsilon) = I_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi i(t) = \begin{cases} 1 & t \in [ih\text{-}\varepsilon, (i+1)h\text{-}\varepsilon) \\ 0 & \text{otherwise,} \end{cases} = I_i,$$

for $i = 1, 2, \dots, m-1$, and

$$\psi_m(t) = \begin{cases} 1 & t \in [T\text{-}\varepsilon,. T) \\ 0 & \text{otherwise} \end{cases} = I_m,$$

with a positive integer value for m and $h = \frac{T}{m}$.
Similar to BPFs, the important properties of εMBPFs are as follows

- Disjointness:

$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t) & i = j, \\ 0 & i \neq j, \end{cases}$$

where $i, j = 0, \dots, m$.

- Orthogonality:

$$\int_0^T \psi_i(t)\psi_j(t)\, dt = h\delta ij$$

where $i, j = 1, \dots, m-1$ and δij is kronecker delta.

- Completeness:

$$\int_0^T f^2(t)dt = \sum_{i=0}^{\infty} fi^2 \|\psi_i(t)\|^2$$

where

$$fi = \frac{1}{\Delta(I_i)} \int_0^T f(t)\psi i(t)dt \qquad (2)$$

and $\Delta(Ii)$ is length of interval Ii.
Rewriting Eq. (2) in the vector form we have

$$f(t) \simeq \sum_{i=0}^{m} fi\psi i(t) = F^T\Psi(t) = \Psi^T(t)F,$$

in which

$F = (f0\ f1\ \dots\ fm)^T$ and $\Psi(t) = (\psi_0(t)\ \psi_1(t)\ \dots\ \psi_m(t))^T$.
Moreover, any two dimensional function $(s, t) \in L^2([0, T_1) \times [0, T_2))$ can be expanded with respect to εMBPFs such as

$$k(s, t) = \Psi^T(s)K\Phi(t) = \Phi^T(t)K^T\Psi(s),$$

where $\Psi(s)$ and $\Phi(t)$ are m_1 and m_2 dimensional εMBPFs vectors respectively, and $K = (kij)$, $i = 0,1, \dots, m_1$, $j = 0,1, \dots, m_2$ is the $m_1 \times m_2$ ε modified block pulse coefficient matrix with

$$kij = \frac{1}{\Delta(Ii)\Delta(Ij)} \int_0^{T_1}\int_0^{T_2} k(s,t)\ \Psi_i(s)\Phi j(t)dtds,$$

For convenience, we put $m_1 = m_2 = m$.
With defining $\Psi_{m+1}(t) = (\psi_0(t)\ \psi_1(t)\ \dots\ \psi_m(t))^T$, we have

$$\Psi_{(m+1)}(t)\Psi_{m+1}^T(t) = \begin{pmatrix} \psi_0(t) & 0 & \cdots & 0 \\ 0 & \psi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_m(t) \end{pmatrix}_{(m+1)x(m+1)},$$

Furthermore,

$$\Psi_{m+1}^T(t)\Psi_{m+1}(t) = 1,$$

And

$$\Psi_{m+1}(t)\Psi_{m+1}^T(t)F = D_F\ \Psi_{m+1}(t)$$

where DF usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_0\ f_1 \dots f_m)^T$.
Similar to BPFs,

$$\int_0^t \Psi_{m+1}(s)ds \simeq Q\Psi_{m+1}(t),$$

where the integration operational matrix Q of εMBPFs is given by

$$Q = \begin{pmatrix} \frac{h\text{-}\varepsilon}{2} & h\text{-}\varepsilon & \cdots & h\text{-}\varepsilon \\ 0 & \frac{h}{2} & \cdots & h \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\varepsilon}{2} \end{pmatrix}_{(m+1)x(m+1)}$$

So, the integral of every function (t) can be approximated as follows

$$\int_0^t f(s)\, ds \simeq \int_0^t F^T\ \Psi_{m+1}(s)ds \simeq F^TQ\Psi_{m+1}(t).$$

The Itô integral of εMBPFs $\psi_i(t)$ can be computed as follows:

$$\int_0^t \psi_0(s)dB(s) = \begin{cases} B(t)\text{-}B(0) & 0 \le t < h\text{-}\varepsilon, \\ B(h\text{-}\varepsilon)\text{-}B(0) & h\text{-}\varepsilon \le t < T, \end{cases}$$

$$\int_0^t \psi_i(s)dB(S) = \begin{cases} 0 & 0 \le t < ih\text{-}\varepsilon, \\ B(t)\text{-}B(ih\text{-}\varepsilon) & ih\text{-}\varepsilon \le t < (i+1)h\text{-}\varepsilon, \\ B((i+1)h\text{-}\varepsilon) & (i+1)h\text{-}\varepsilon \le t < T, \end{cases}$$

for $i = 1,2, \dots, m$, and

$$\int_0^t \psi_m(s)dB(s) = \begin{cases} 0 & 0 \le t < T\text{-}\varepsilon \\ B(t)\text{-}B(T\text{-}\varepsilon) & T\text{-}\varepsilon \le t < T \end{cases}$$

Since $B(t) - B(ih - \varepsilon)$ equals to $B((i + 0.5)h - \varepsilon) - B(ih - \varepsilon)$, at midpoint of $[ih - \varepsilon, (i + 1)h - \varepsilon)$. We can approximate $(t) - (0)$ by $B\left(\frac{(h\text{-}\varepsilon)}{2}\right)$ in $\psi_0(t)$ at midpoint of $[0, h - \varepsilon)$ and $(t) - (T - \varepsilon)$ by $B\left(T\text{-}\left(\frac{\varepsilon}{2}\right)\right) - B(T - \varepsilon)$ in $\psi_m(t)$ at midpoint of $[T - \varepsilon, T)$. It has the vector form is given by

$$\int_0^t \psi_0(s)dB(s) \simeq \left(B\left(\frac{h\text{-}\varepsilon}{2}\right) B(h\text{-}\varepsilon) \dots B(h\text{-}\varepsilon)\right)\psi(t),$$

$$\int_0^t \psi_i(s)dB(s) \simeq \big(0\ 0 \dots 0\ B((i+0.5)h\text{-}\varepsilon)\text{-}B(ih\text{-}\varepsilon)\ B((i+1)h\text{-}\varepsilon)\text{-}B(ih\text{-}\varepsilon) \dots B((i+1)h\text{-}\varepsilon)\text{-}B(ih\text{-}\varepsilon)\big)\Psi(t),$$

in which the $(i + 1)$ th component is $B((i + 0.5)h - \varepsilon) - B(ih - \varepsilon)$,

$$\int_0^t \psi_m(s)dB(s) \simeq \left(0\ 0 \dots B\left(T\text{-}\frac{\varepsilon}{2}\right)\text{-}B(T\text{-}\varepsilon)\right)\Psi(t).$$

Therefore,

$$\int_0^t \Psi(s)dB(s) \simeq Qs\Psi(t),$$

where stochastic integration operational matrix is given by

$$Q_s = \begin{pmatrix} B\left(\frac{h\text{-}\varepsilon}{2}\right) & B(h\text{-}\varepsilon) & B(h\text{-}\varepsilon) & \cdots & B(h\text{-}\varepsilon) \\ 0 & B\left(\frac{3h}{2}\text{-}\varepsilon\right)\text{-}B(h\text{-}\varepsilon) & B(2h\text{-}\varepsilon)\text{-}B(h\text{-}\varepsilon) & \cdots & B(2h\text{-}\varepsilon)\text{-}B(h\text{-}\varepsilon) \\ 0 & 0 & B\left(\frac{5h}{2}\text{-}\varepsilon\right)\text{-}B(2h\text{-}\varepsilon) & \cdots & B(3h\text{-}\varepsilon)\text{-}B(2h\text{-}\varepsilon) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B\left(T\text{-}\frac{\varepsilon}{2}\right)\text{-}B(T\text{-}\varepsilon) \end{pmatrix}_{(m+1)x(m+1)}.$$

So, the Itô integral of every function (t) can be approximated as follows

$$\int_0^t f(s)dB(s) \simeq \int_0^t F^T\Psi(s)dB(s) \simeq F^TQ_s\Psi(t)\cdot$$

RESULTS AND DISCUSSION

Solving Stochastic Volterra-Fredholm **Integral Equations by Using $\boldsymbol{\varepsilon}$ Modified Block Pulse Functions**

We consider following linear stochastic Volterra-Fredholm integral equation

$$X(t) = f(t) + \int_\alpha^\beta k_1(s,t)X(s)ds + \int_0^t k_2(s,t)X(s)ds + \int_0^t k_3(s,t)X(s)dB(s), t \in [0,T). \quad (3)$$

We approximate functions (t), (t), $k_1(s, t)$, $k_2(s, t)$, and $k_3(s, t)$ by εMBPFs:

$(t) \simeq W^T\Phi(t) = \Phi^T(t)W$, $f(t) \simeq F^T\Phi(t) = \Phi^T(t)F$,
$k_1(s, t) \simeq \Psi^T(s)K_1\Phi(t) = \Phi^T(t)K_1^T\Psi(s)$,
$k_2(s, t) \simeq \Psi^T(s)K_2\Phi(t) = \Phi^T(t)K_2^T\Psi(s)$,
$k_3(s, t) \simeq \Psi^T(s)K_3\Phi(t) = \Phi^T(t)K_3^T\Psi(s)$,

In the above approximates, W and F are stochastic modified block pulse coefficients vector, and $K1$, $K2$, and $K3$ are stochastic modified block pulse coefficients matrix. Substituting above approximation in Eq. (3), we get

$$W^T\Phi(t) \simeq F^T\Phi(t) + W^T\left(\int_\alpha^\beta \Psi(s)\,\Psi^T(s)ds\right)K_1\Phi(t) + W^T\left(\int_0^t \Psi(s)\Psi^T(s)ds\right)K_2\Phi(t) + W^T\left(\int_0^t \Psi(s)\Psi^T(s)dB(s)\right)K_3\Phi(t). \quad (4).$$

Let K_j^i be the ith row of the constant matrices K_j, $j = 1,2,3$. R^i be the ith row of the integration operational matrix Q, R_s^i be the ith row of the stochastic integration operational matrix Q_s, $D_{K_j^i}$ be diagonal matrices with K_j^i as its diagonal entries. By the relation

$\int_\alpha^\beta \Psi(s)\Psi^T(s)ds\, hI_{(m_1+1)x(m_2+1)}$ and assuming $m_1 = m_2 = m$, we have

$$\left(\int_\alpha^\beta \Psi(s)\Psi^T(s)ds\right)K_1\Phi(t) = hIK_1\Phi(t)$$

$$= B_1\Phi(t)\,, \qquad (5)$$

where $B_1 = hIK_1 = hK_1$. Furthermore,

$$\left(\int_0^t \Psi(s)\Psi^T(s)ds\right)K_2\Phi(t)$$

$$= \left(\int_0^t \Phi(s)\Phi^T(s)ds\right)K_2\Phi(t)$$

$$= \begin{pmatrix} R^0\Phi(t) & 0 & \cdots & 0 \\ 0 & R^1\Phi(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R^m\Phi(t) \end{pmatrix}\begin{pmatrix} K_2^0 \\ K_2^1 \\ \vdots \\ K_2^m \end{pmatrix}\Phi(t)$$

$$= \begin{pmatrix} R^0\Phi(t)K_2^0\Phi(t) \\ R^1\Phi(t)K_2^1\Phi(t) \\ \vdots \\ R^m\Phi(t)K_2^m\Phi(t) \end{pmatrix}$$

$$= \begin{pmatrix} R^0\Phi(t)\Phi^T(t){K_2^0}^T \\ R^1\Phi(t)\Phi^T(t){K_2^1}^T \\ \vdots \\ R^m\Phi(t)\Phi^T(t){K_2^m}^T \end{pmatrix}$$

$$= \begin{pmatrix} R^0D_{K_2^0} \\ R^1D_{K_2^1} \\ \vdots \\ R^mD_{K_2^m} \end{pmatrix}\Phi(t) = B_2\Phi(t), \qquad (6)$$

where

$$B_2 = \begin{pmatrix} k_{00}\left(\frac{h-\varepsilon}{2}\right) & k_{01}(h-\varepsilon) & \cdots & k_{0m}(h-\varepsilon) \\ 0 & k_{11}\left(\frac{h}{2}\right) & \cdots & k_{1m}(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{mm}\left(\frac{\varepsilon}{2}\right) \end{pmatrix}_{(m+1)x(m+1)}$$

and then in the same way obtained

$$\left(\int_0^t \Psi(s)\Psi^T(s)dB(s)\right)K_3\Phi(t) = B_3\Phi(t), \qquad (7)$$

Where

$$B_3 = \begin{pmatrix} k_{00}B\left(\frac{h-\varepsilon}{2}\right) & k_{01}B(h-\varepsilon) & \cdots & k_{0m}B(h-\varepsilon) \\ 0 & k_{11}\left(B\left(\frac{3h}{2}-\varepsilon\right)-B(h-\varepsilon)\right) & \cdots & \left(B(2h-\varepsilon)-B(h-\varepsilon)\right) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{mm}\left(B\left(T-\frac{\varepsilon}{2}\right)-B(T-\varepsilon)\right) \end{pmatrix}_{(m+1)x(m+1)}$$

With substituting (5), (6), and (7) in (4), we get $W^T\Phi(t) \simeq F^T\Phi(t) + W^TB_1\Phi(t) + W^TB_2\Phi(t) + W^TB_3\Phi(t)$. Then, $W^T(I-B_1-B_2-B_3) \simeq F^T$. So, by setting $N = (I-B_1-B_2-B_3)^T$ and replacing $\simeq$ by =, we have $NW = F$.

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficient of the unknown stochastic processes (t).

Error Analysis

In the following theorems, for simplicity we assume $T = 1$ and $h = \frac{1}{m}$.

Theorem 1. If $\hat{f}_m(t) = \sum_{i=0}^m f_i\, \psi_i(t)$ and $f_i = \frac{1}{\Delta(I_i)}\int_0^1 f(t)\psi_i(t)dt$, $i = 0, \dots, m$ then:

i. $\delta = \int_0^1 (f(t)-\sum_{i=0}^m f_i\, \psi_i(t))^2dt$, achieves its minimum value.

ii. $\{\hat{f}_m(t)\}$ *approaches* $f(t)$ *pointwise*.

iii $\int_0^1 f^2(t)dt = \sum_{i=0}^\infty {f_i}^2\, \|\psi_i\|^2$.

[Proof]. Proof is like similar theorem in (Jiang, 1992) but intervals of integration have to redefine as I_i, $i = 0, \dots, m$ in(3.1). ■

Theorem 2. *Assume*:

i. $f(t)$ *is continuous and differentiable in* $[-h, 1+h]$ *with bounded derivative, that is* $|f'(t)| < M$.

ii. $\hat{f}_{\frac{ih}{k}}(t), i = 0, \dots, k-1$ *are correspondingly BPFs*, $\frac{h}{k}$ *MBPFs* ... , $\frac{(k-1)h}{k}$ *MBPFs expansions of* $f(t)$ *base on* $m+1$ *εMBPFs over interval* [0,1).

iii. $\bar{f}(t) = \frac{1}{k}\sum_{i=0}^{k-1}\hat{f}_{\frac{ih}{k}}(t)$.

Then

$\left\|f(t)-\hat{f}_{\frac{ih}{k}}(t)\right\| = O(h)$, and $\|f(t)-\bar{f}(t)\| = \mathcal{O}\left(\frac{h}{k}\right)$ in $[h, 1-h]$.

[Proof]. Trapezoidal rule for integral is

$$\int_a^b f(t)dt = \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3f''(\eta)}{12}$$

$$= \frac{b-a}{2}(f(a)+f(b)) + E, \eta \in [a,b], \qquad (8)$$

where E is error of integration. Suppose $t_i = \frac{i}{m} = ih$ and $I_i = [t_{i-1}, t_i]$. The representation error when (t) is represented by a series of BPFs over every subinterval

$\left[t_i, t_i + \frac{h}{k}\right], i = 0, \dots, m\text{-}1$ is

$e_i(t) = f(t)\text{-}f_i\psi_i(t) = f(t)\text{-}f_i$, where $f_i = \frac{i}{h}\int_{ih}^{(i+1)h} f(t)dt$.

From (8), $f_i = \frac{1}{2}\left(f(t_i) + f(t_i + h)\right) + E$. It is obvious that if $(t) = C$ (constant), then $e_i(t) = 0$. So, this error is computed for $(t) = t$ in interval $\left[t_i, t_i + \frac{h}{k}\right], i = 1, \dots, m\text{-}1$. For this function $E = 0$, so

$$e_i(t)_{\left[t_i, t_i+\frac{h}{k}\right]} = |t\text{-}f_i| = \left|t\text{-}\frac{t_i + t_{i+1}}{2}\right| = \left|t\text{-}\left(t_i + \frac{h}{2}\right)\right| \leq \frac{h}{2}.$$

Then this error with BPFs is $\frac{h}{2}M$. Similarly, the error when (t) is represented in a series of εMBPFs over every subinterval $\left[t_i, t_i + \frac{h}{k}\right]$ is

$$e_i(t)_{\left[t_i, t_i+\frac{h}{k}\right]} = \left|t\text{-}\left(\frac{\sum_{j\text{-}0}^{k\text{-}1}\left(t_i\text{-}\left(\frac{jh}{k}\right) + t_{i+1}\text{-}\left(\frac{jh}{k}\right)\right)}{2k}\right)\right|$$

$$= \left|t\text{-}\left(\frac{\sum_{j\text{-}0}^{k\text{-}1}\left(t_i\text{-}\left(\frac{jh}{k}\right) + t_i + h\text{-}\left(\frac{jh}{k}\right)\right)}{2k}\right)\right|$$

$$= \left|t\text{-}\left(t_i + \frac{h}{2}\right)\text{-}\frac{(k\text{-}1)h}{2k}\right|$$

$$\leq \frac{h}{2k}.$$

So, the error with εMBPFs is $\frac{h}{2k}M$. For I_0 in $\left[0, \frac{h}{k}\right]$ we have

$$e_i(t)_{\left[0,\frac{h}{k}\right]} = \left|t\text{-}\sum_{j=0}^{k\text{-}1}\frac{h\text{-}\left(\frac{jh}{k}\right)}{2k}\right|$$

$$= \left|t\text{-}\left(\frac{h}{2}\text{-}\frac{(k\text{-}1)h}{4k}\right)\right|$$

$$= \left|t\text{-}\left(\frac{h}{4} + \frac{h}{4k}\right)\right|$$

$$= O\left(\frac{h}{4}\right).$$

So, the error is $\mathcal{O}\left(\frac{h}{4}\right)$ also for I_n.

Now,

$$\|e_i(t)\|^2 = \int_{t_i}^{t_i+\frac{h}{k}} |e_i(t)|^2 dt$$

$$= \int_{t_i}^{t_i+\frac{h}{k}} \frac{h^2}{4k^2}M^2 dt$$

$$= \frac{h^3}{4k^3}M^2,$$

$$\|e\|^2 = \int_0^1 e^2(t)dt$$

$$= \int_0^1 \left(\sum_{i=1}^{m}\sum_{j=0}^{k\text{-}1} e_i(t)\right)^2 dt$$

$$= \sum_{i=1}^{m}\sum_{j=0}^{k\text{-}1}\int_0^1 {e_i}^2(t)dt$$

$$= \sum_{i=1}^{m}\sum_{j=0}^{k\text{-}1}\|e_i(t)\|^2$$

$$= \frac{1}{h}.k.\frac{h^3}{4k^3}M^2$$

$$= \frac{h^2}{4k^2}M^2.$$

We define the representation error between (s, t) and its 2D-εMBPFs expansion f_{ij} over every subregion D_{ij}, is defined as

$$e_{ij}(s,t) = f(s,t)\text{-}f_{ij},$$

where $D_{ij} \coloneqq \left\{(s,t)\middle| t_i \leq s \leq t_i + \frac{h}{k}, t_j \leq t \leq t_j + \frac{h}{k}\right\}$.

With Taylor's expansion and similarity to the above discussion,

$$\|e(s,t)\| = \frac{h}{2k}M. \blacksquare$$

Theorem 3. *Assume that*

i. $P(\omega \in \Omega : \|u(\omega,t)\| < C) = 1$.

ii. $\|k_i\| < C, i = 1,2$.

Then

$\sup\left(E(\|(u\text{-}\bar{u})\|)^2\right)^{\frac{1}{2}} = O\left(\frac{h}{k}\right),\ t \in [h, 1\text{-}h]$.

[Proof]. For a complete proof see (Maleknejad, 2014).

Numerical Example

Consider the following linear stochastic Volterra-Fredholm integral equation,

$$X(t) = f(t) + \int_0^1 \cos(s,t)X(s)ds + \int_0^t (s,t)\,X(s)ds + \int_0^t e^{\text{-}3(s+t)}X(s)dB(s), \quad s,t \in [0,1),$$

with $f(t) = t^2 + \sin(1+t)\text{-}2\cos(1+t)\text{-}2\sin(t)\text{-}\frac{7t^4}{12} + \frac{1}{40}B(t)$.

The approximation solution are shown in Table 1. The curves in Fig. 1 represents a trajectory of the approximate solution computed by the presented method.

Table 1. The approximate solutions of Example for m = 129, 257.

t	m =129	m=257
0	-0.0008668	-0.01731
0.1	0.0116	0.0005096
0.2	0.03985	0.02713
0.3	0.08992	0.07726
0.4	0.158	0.1351
0.5	0.2579	0.2376
0.6	0.3609	0.3655
0.7	0.4905	0.4717
0.8	0.6333	0.5952
0.9	0.803	0.7455
1	0.9862	0.9331

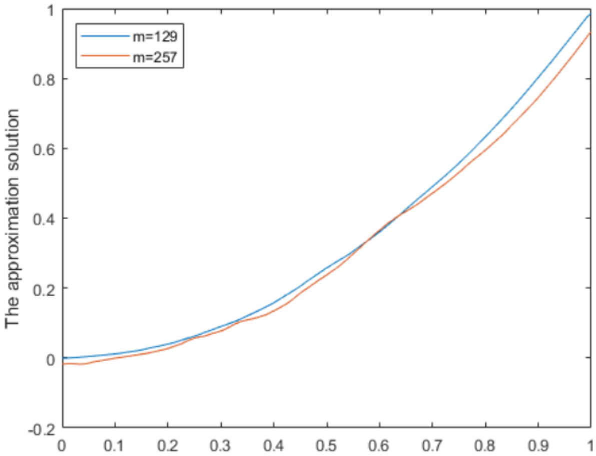

Figure 1. The trajectory of the approximation solution of example.

CONCLUSIONS

Because it is almost impossible to find the exact solution of SV-FIEs, it would be convenient to determine its numerical solution based on stochastic numerical analysis. The modified block pulse functions and their stochasticintegration operational matrix a new computational method is proposed for solving the linear SV-FIEs is simple and effective. Its applicability and accuracy of the proposed method was checked on example. The results show that the approximate solutions of the proposed method have a good of efficiency and accuracy.

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