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CHARACTERIZATION OF CAUCHY-SCHWARZ INEQUALITY WITH TAPIA SEMI-INNER PRODUCT ON COMPLEX INNER PRODUCT SPACES

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Abstract

Tapia semi-inner product is defined on a normed space. If a normed space is constructed by an inner product, then the Tapia semi-inner product is equivalent to that inner product. This research constructs a refinement of the Cauchy-Schwarz inequality using the Tapia semi-inner product. Furthermore, the characterization of the Cauchy-Schwarz inequality is extended to complex inner product spaces with specific adjustments. The results provide a new perspective on triangle inequality characterization in complex domains.

Keywords: Cauchy-Schwarz Inequality; Complex Inner Product Space; Normed Space; Tapia Semi-Inner Product; Triangle Inequality.

1. INTRODUCTION

Norm is a fundamental concept in functional analysis, allowing for the definition of the Tapia semi-inner product within normed spaces. While every inner product space is a normed space by defining the norm through the inner product, the converse does not necessarily hold. Consequently, a Tapia semi-inner product defined on a general normed space is not always an inner product unless the norm is specifically induced by an inner product. The exploration of mapping properties within these spaces has been previously discussed, particularly focusing on the characteristics of semi-inner products in real normed spaces (Futhona, 2021).

The Cauchy-Schwarz inequality is one of the properties in an inner product space. In a normed space, this Cauchy-Schwarz inequality is refined with the Tapia semi-inner product. Furthermore, this characterization is brought into the complex inner product space. Specifically, if a complex inner product is defined, then using the final result, an inequality in another form will be obtained. This also provides an example of the application of the Cauchy-Schwarz inequality characterization in a complex inner product space. The aim of this research is to construct a refinement of the Cauchy-Schwarz inequality and its application in complex spaces.

2. MATERIALS AND METHODS

This section provides the fundamental mathematical definitions and theorems that form the basis of this research. The discussion begins with the basic concepts of normed spaces and inner product spaces as the primary environment for the analysis. These established theories are essential for understanding the construction of the Tapia semi-inner product and the subsequent refinement of the Cauchy-Schwarz inequality. The following is the definition of a normed space as discussed by Darmawijaya (2007).

Definition 2.1. Let X be a vector space over the field $\mathbb{C}$. A function $\|\cdot\|: X \to \mathbb{R}$ is called a norm on X if for every $x, y \in X$, $\alpha \in \mathbb{R}$ satisfies:

(N1) $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$

(N2) $\|\alpha x\| = |\alpha|\|x\|$

(N3) $\|x + y\| \leq \|x\| + \|y\|$

Beyond the structure of normed spaces, it is crucial to establish the properties of inner product spaces, particularly in the complex domain. The following definition outlines the axiomatic requirements for a function to be recognized as an inner product, which allows for the representation of geometric concepts such as angles and lengths. This framework is pivotal for the transition from general normed spaces to the more specific inner product spaces used in this study. The definition of an inner product space as written by Heil (2018) is provided.

Definition 2.2. Let X be a vector space over the field $\mathbb{C}$. A function $\langle\cdot,\cdot\rangle: X \times X \to \mathbb{C}$ is called an inner product on X if for every $x, y, z \in X$, $a, b \in \mathbb{R}$ satisfies:

(IP1) $\langle x, x\rangle \geq 0$

(IP2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(IP3) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

(IP4) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

Furthermore, a vector space X equipped with an inner product is called an inner product space, denoted by $(X, \langle\cdot,\cdot\rangle)$. If the function only satisfies axioms (IP1), (IP2) and (IP3), it is referred to as a semi-inner product, and the vector space X equipped with it is called a semi-inner product space.

The following theorem describes the relationship regarding a normed space constructed by an inner product, as discussed by Berberian (1999). This theorem is fundamental because it guarantees that every inner product space can be treated as a normed space by inducing a norm from its inner product. Such a construction allows for the bridge between the geometric properties of inner products and the analytical properties of normed spaces, which is essential for defining the Tapia semi-inner product later in this study.

Theorem 2.3. Every inner product space $(X, \langle\cdot,\cdot\rangle)$ is a normed space.

Proof. Define $\|\cdot\|: X \to \mathbb{R}$, where $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in X$. Then for every $x, y \in X$ and scalar $\alpha \in \mathbb{C}$ satisfies:

i) Since $\langle x, x\rangle \geq 0$ then $\|x\| = \sqrt{\langle x, x\rangle} \geq 0$.
ii) Since $\langle x, x\rangle = 0 \Leftrightarrow x = 0$ then $\|x\| = \sqrt{\langle x, x\rangle} = 0 \Leftrightarrow x = 0$.
iii) Since $\sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\langle x, \alpha x\rangle} = \sqrt{\alpha\overline{\langle \alpha x, x\rangle}} = \sqrt{\alpha\bar{\alpha}\overline{\langle x, x\rangle}} = \sqrt{\alpha^2\overline{\langle x, x\rangle}} = |\alpha|\sqrt{\overline{\langle x, x\rangle}}$ then $\|\alpha x\| = |\alpha|\sqrt{\langle x, x\rangle} = |\alpha|\|x\|$.
iv) Since

$$\begin{aligned}\langle x + y, x + y\rangle &= \langle x, x + y\rangle + \langle y, x + y\rangle \\ &= \overline{\langle x + y, x\rangle} + \overline{\langle x + y, y\rangle} \\ &= \overline{\langle x, x\rangle + \langle y, x\rangle} + \overline{\langle x, y\rangle + \langle y, y\rangle} \\ &= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2\end{aligned}$$

and $Re\langle x, y\rangle \leq |\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle} = \|x\|\|y\|$, then

$$\|x + y\|^2 = \langle x + y, x + y\rangle \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

Thus, $\|x + y\| \leq \|x\| + \|y\|$. ■

3. RESULTS AND DISCUSSION

The core of this analysis lies in the utilization of the Tapia semi-inner product, which serves as a functional tool to approximate inner product properties in general normed spaces. This mapping is particularly useful because it relies solely on the norm of the space, providing a directional derivative of the squared norm. To establish a formal framework for our refinement of the Cauchy-Schwarz inequality, we first state the following definition as the primary foundation for the subsequent derivations.

Definition 3.1. Let X be a normed space. The Tapia semi-inner product function $[x,y]_T: X \times X \to \mathbb{R}$ is defined as:

$$[x,y]_T = \lim_{t\to 0^+} \frac{1}{2t}(\|x+ty\|^2 - \|x\|^2) \tag{1}$$

Furthermore, $(X,[x,y]_T)$ is called the Tapia semi-inner product space.

By employing the properties of the Tapia semi-inner product, it is possible to derive a more precise bound for the triangle inequality in real normed spaces. Furthermore, a refinement of the Cauchy-Schwarz inequality using the Tapia semi-inner product has been previously investigated by Minculete (2020), as presented in the following theorem.

Theorem 3.2. If $(X,\|\cdot\|)$ is a real normed space, then for every $x,y \in X$ satisfy:

$$0 \le \|x\|(\|x\| + \|y\| - \|x+y\|) \le \|x\|\|y\| - [x,y]_T. \tag{2}$$

Proof. Let $x,y \in X$. Since $\|x+y\| \le \|x\| + \|y\|$ then $0 \le \|x\| + \|y\| - \|x+y\|$. Multiplying by $\|x\|$ gives

$$0 \le \|x\|(\|x\| + \|y\| - \|x+y\|). \tag{3}$$

Based on the results by Minculete (2020), specifically $\min\{a,b\}(\|x\| + \|y\| - \|x+y\|) \le a\|x\| + b\|y\| - \|ax+by\|$ for all $x,y \in X$ and $a,b > 0$, by substituting $a = 1$ and $b = t > 0$, we obtain

$$\min\{1,t\}(\|x\| + \|y\| - \|x+y\|) \le \|x\| + t\|y\| - \|x+ty\|$$
$$\Leftrightarrow \frac{1}{\mathrm{t}}\min\{1,t\}(\|x\| + \|y\| - \|x+y\|) \le \frac{\|x\|}{t} + \|y\| - \frac{\|x+ty\|}{t}$$
$$\Leftrightarrow \frac{1}{\mathrm{t}}\min\{1,t\}(\|x\| + \|y\| - \|x+y\|) \le \|y\| - \frac{\|x+ty\| - \|x\|}{t}.$$

As $t \to 0^+$, we obtain $\min\{1,t\} = t$. This implies

$$\begin{aligned}\|x\| + \|y\| - \|x+y\| &\le \|y\| - \lim_{t\to 0^+}\frac{\|x+ty\| - \|x\|}{t} \\ &= \|y\| - \lim_{t\to 0^+}\frac{(\|x+ty\| - \|x\|)(\|x+ty\| + \|x\|)}{t(\|x+ty\| + \|x\|)} \\ &= \|y\| - \lim_{t\to 0^+}\frac{\|x+ty\|^2 - \|x\|^2}{t}\frac{1}{2\|x\|}\end{aligned}$$

Regarding (1) and multiplying by $\|x\|$ gives

$$\|x\|(\|x\| + \|y\| - \|x+y\|) \le \|x\|\|y\| - [x,y]_T \tag{4}$$

From (3) and (4) we get (4). ■

Furthermore, the characterization of inequality (2), which holds in general normed spaces, is extended to the context of complex inner product spaces in the following theorem. This extension is crucial as it addresses the complexities introduced by the imaginary components of the scalars, ensuring that the refinement remains robust even in more generalized mathematical structures. By bridging these concepts, we can better understand how the real part of the inner product preserves the geometric integrity of the Cauchy-Schwarz refinement in complex domains.

Theorem 3.3. Let $(X,\langle\cdot,\cdot\rangle)$ be a complex inner product space. If $\|\cdot\|$ is the induced norm, then for every $x,y \in X$ satisfy:

$$0 \le \|x\|(\|x\| + \|y\| - \|x+y\|) \le \|x\|\|y\| - Re\langle x,y\rangle. \tag{5}$$

Proof. Since $\|x+y\|^2 = \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2$, multiplying by $\|x\|^2$ gives

$$\|x\|^2\|x+y\|^2 = \|x\|^4 + 2\|x\|^2 Re\langle x,y\rangle + (\|x\|\|y\|)^2.$$

Since $Re\langle x,y\rangle \le |\langle x,y\rangle| \le \|x\|\|y\|$, then

$$(\|x\|\|x+y\|)^2 \ge \|x\|^4 + 2\|x\|^2 Re\langle x,y\rangle + (Re\langle x,y\rangle)^2 = (\|x\|^2 + Re\langle x,y\rangle)^2.$$

Furthermore,

$$\|x\|^2 + Re\langle x,y\rangle \le \|x\|\|x+y\| \Leftrightarrow \|x\|^2 - \|x\|\|x+y\| \le -Re\langle x,y\rangle$$

Summing with $\|x\|\|y\|$ yields $\|x\|^2 + \|x\|\|y\| - \|x\|\|x+y\| \le \|x\|\|y\| - Re\langle x,y\rangle$, which equivalent to

$$\|x\|(\|x\| + \|y\| - \|x+y\|) \le \|x\|\|y\| - Re\langle x,y\rangle.$$

Based on the properties of the norm, inequality (3) also holds for complex normed spaces. Therefore, we obtain inequality (5). ■

A further consequence of the aforementioned characterizations is the ability to construct a stronger version of the triangle inequality that incorporates the modulus of the inner product. By manipulating the expansion of the squared norm and applying the Cauchy-Schwarz property, we can derive an inequality that provides a tighter lower bound for the norm difference. This result is formally presented in the following theorem, which offers a new perspective on the classical inequality.

Theorem 3.4. Let $(X, \langle\cdot,\cdot\rangle)$ be a complex inner product space and $x, y \in X$. If $\|\cdot\|$ is the induced norm, then for $x \neq 0$ or $y \neq 0$ satisfy:

$$\|x\| + \|y\| - \|x+y\| \ge \frac{\|x\|\|y\| - |\langle x,y\rangle|}{\|x\| + \|y\|}. \tag{6}$$

Proof. Since $\|x+y\|^2 = \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2$ and $Re\langle x,y\rangle \le |\langle x,y\rangle|$, then

$$\begin{aligned} &\|x+y\|^2 = (\|x\| + \|y\|)^2 - 2\|x\|\|y\| + 2Re\langle x,y\rangle \\ \Leftrightarrow\ & 2\|x\|\|y\| - 2|\langle x,y\rangle| = (\|x\| + \|y\|)^2 - \|x+y\|^2 \\ \Leftrightarrow\ & 2\|x\|\|y\| - 2|\langle x,y\rangle| \le (\|x\| + \|y\|)^2 - \|x+y\|^2. \end{aligned}$$

Since $\|x+y\| \le \|x\| + \|y\|$, we obtain

$$\begin{aligned} 2(\|x\|\|y\| - |\langle x,y\rangle|) &\le (\|x\| + \|y\| + \|x+y\|)(\|x\| + \|y\| - \|x+y\|) \\ &\le (\|x\| + \|y\| + \|x\| + \|y\|)(\|x\| + \|y\| - \|x+y\|) \\ &= 2(\|x\| + \|y\|)(\|x\| + \|y\| - \|x+y\|). \end{aligned}$$

Therefore, inequality (6) holds. ■

While the proof of Theorem 3.4 was established by Minculete (2020), a concrete numerical implementation of the theorem has not yet been presented. Therefore, the following example is provided to complement their theoretical findings and demonstrate the practical application of the theorem. By examining this result within a specific mathematical framework, we can verify the validity of the refined inequality and illustrate its behavior in a computational context.

Example 3.5. Let $X = \mathbb{C}^n$ be a complex inner product space with $\langle x,y\rangle = \sum_{k=1}^{n} x_k\overline{y_k}$. This satisfies IP1-IP4. Define $\|\cdot\|$ as the induced norm, $\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{\sum_{k=1}^{n} x_k\overline{x_k}}$. Applying Theorem 3.4, for $x = (x_1, x_2, \dots) \neq 0$ or $y = (y_1, y_2, \dots) \neq 0$ we obtain

$$\sqrt{\sum_{k=1}^{n} x_k\overline{x_k}} + \sqrt{\sum_{k=1}^{n} y_k\overline{y_k}} - \sqrt{\sum_{k=1}^{n}(x_k + y_k)\overline{(x_k + y_k)}} \ge \frac{\sqrt{\sum_{k=1}^{n} x_k\overline{x_k}}\sqrt{\sum_{k=1}^{n} y_k\overline{y_k}} - \left|\sum_{k=1}^{n} x_k\overline{y_k}\right|}{\sqrt{\sum_{k=1}^{n} x_k\overline{x_k}} + \sqrt{\sum_{k=1}^{n} y_k\overline{y_k}}}.$$

Taking $a = \sqrt{\sum_{k=1}^{n} x_k\overline{x_k}}$, $b = \sqrt{\sum_{k=1}^{n} y_k\overline{y_k}}$, $c = \sqrt{\sum_{k=1}^{n}(x_k + y_k)\overline{(x_k + y_k)}}$, and $d = \left|\sum_{k=1}^{n} x_k\overline{y_k}\right|$, satisfy

$$\begin{aligned} a + b - c \ge \frac{ab - |d|}{a+b} &\Leftrightarrow (a+b)(a+b-c) \ge ab - |d| \\ &\Leftrightarrow a^2 + 2ab - ac - bc + b^2 \ge ab - |d| \\ &\Leftrightarrow a^2 - c(a+b) + b^2 \ge -ab - |d| \end{aligned}$$

Therefore, we get a new inequality form:

$$\sum_{k=1}^{n} x_k\overline{x_k} - \sqrt{\sum_{k=1}^{n}(x_k + y_k)\overline{(x_k + y_k)}}\left(\sqrt{\sum_{k=1}^{n} x_k\overline{x_k}} + \sqrt{\sum_{k=1}^{n} y_k\overline{y_k}}\right) + \sum_{k=1}^{n} y_k\overline{y_k}$$

$$\geq -\sqrt{\sum_{k=1}^{n} x_k\overline{x_k}}\sqrt{\sum_{k=1}^{n} y_k\overline{y_k}} - \left|\sum_{k=1}^{n} x_k\overline{y_k}\right|.$$

4. CONCLUSION

The characterization of the triangle and Cauchy-Schwarz inequalities using the Tapia semi-inner product provides a robust framework for refinements in complex inner product spaces. The implementation of Theorem 3.4 through specific examples in $\mathbb{C}^n$ confirms the validity of the newly derived inequality forms.

5. REFERENCES

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